THERMALLY SUSTAINED PRESSURE
OSCILIATIONS IN LIQUID HELIUM
APPARATUS

Author: Richard John Howson

Thesis Supervisor: Joseph L. Smith,
Title: Associate Professor of
Mechanical Engineering

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by

RICHARD JOHN HOWSON

LIEUTENANT, UNITED STATES NAVY

B.A., DePauw University

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Signature of Author: ______________________________
Department of Naval Architecture and Marine Engineering, May 17, 1968

Certified by: ______________________________
Thesis Supervisor

Certified by: ______________________________
Department Reader

Accepted by: ______________________________
Chairman, Departmental Committee on Graduate Students

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ABSTRACT

The phenomenon of thermally sustained pressure oscillations which frequently occur over liquid helium and other liquified gases is modeled and analyzed. The model is a distributed system with a linear temperature gradient. The interaction between the vapor motion and heat transfer is analyzed to derive an expression for the time history of the pressure oscillations.

Cyclic interpretation of the pressure history results in a relationship between four parameters which control the behavior of the oscillations. The four parameters contain information on amplitude of motion, slenderness ratio of the tube, characteristic lengths of the tube, thermal conductivity, specific heat ratio and viscosity of the gas undergoing the oscillations and boundary layer decay.

The effect of changes within the parameters on the theoretical behavior of the oscillating system shows good agreement with the behavior of oscillations in previous experimental apparatus when similiar changes are made.

Thesis Supervisor: Joseph L. Smith, Jr.

Title: Associate Professor of Mechanical Engineering

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Nomenclature

The following nomenclature will be used throughout the text unless otherwise noted.

Symbol	Definition
A	Cross sectional area of tube.
A'	Surface area for heat transfer.
A_w	Surface area for shear stress.
A_1	Constant coefficient.
A_2	Constant Coefficient.
a	Temperature gradient along tube wall.
C	Constant.
c_p	Specific heat at constant pressure.
c_v	Specific heat at constant volume.
D	Viscous drag coefficient.
D_1	Constant coefficient.
D_2	Constant coefficient.
d	Diameter of tube.
k	Thermal conductivity.
L	Length of tube.
L'	Reduced length for shear stress.
l	Length of temperature gradient.
l'	Length of compressible volume.
m	Mass.
N_1	Non-dimensional parameter.
N_2	Non-dimensional parameter.

Symbol	Definition
N_3	Non-dimensional parameter.
n	Angular frequency.
P	Pressure.
$\bar{P}$	Average pressure.
P_∞	Storage vessel pressure.
$\dot{Q}$	Heat transfer rate.
R	Gas constant.
T	Temperature.
T_l	Cold vapor temperature.
T_h	Warm volume temperature.
T_w	Wall surface temperature.
$\bar{T}$	Average temperature.
t	Time.
U	Internal energy.
u	Velocity.
u_o	Maximum velocity amplitude.
V	Volume.
v	Average velocity.
x	Direction along tube or flat plate.
x_m	Thullen's maximum position of inertial element.
x_o	Maximum displacement amplitude.
y	Height above flat plate.
y_m	Maximum height above flat plate.
α	Thermal diffusivity.
γ	Specific heat ratio.
δ	Spacial decay rate of momentum boundary layer.

Symbol	Definition
ν	Kinematic viscosity.
μ	Viscosity.
ξ	Spacial decay rate of thermal boundary layer.
ρ	Density.
τ_w	Shear stress on tube wall.
ϕ	Non-dimensional parameter.
ψ	Non-dimensional parameter.

I. INTRODUCTION

Thermally sustained pressure oscillations are often found in the vapor trapped within cryogenic equipment. Particularly susceptable to this phenomenon are the tubes and piping which penetrate storage containers of liquified gases. These tubes are characterized by being closed at their outside end which is usually at room temperature and open at their other end which is in communication with the cold vapor of the liquified gas. Since it is usually necessary to fill, vent, or measure temperature and pressure within storage vessels by means of a tube or pipe penetration, construction of cryogenic equipment without this configuration is nearly impossible.

Occasionally the oscillations have been used to good advantage for such purposes as vapor mixing or liquid level measurement. However, in most instances, the presence of the pressure oscillations is undesirable. When their pressure amplitudes are sufficiently large, the oscillations can make vapor pressure measurements erroneous. Since the open end of the tube is in free communication with the cold vapor above the liquid, there is a heat pumping action to the low temperature region which greatly increases the boil-off rate of the liquified gas within the container.

Encounters with the phenomenon in cryogenic apparatus is well documented in cryogenic engineering literature.

The phenomenon is most readily observable over liquid helium. Investigations by Ditmars and Furukawa (3) and Thullen (7) have shown that the pressure oscillations may occur over liquid hydrogen or liquid nitrogen, although their detection is somewhat more difficult than over liquid helium. Thullen (7) gives a rather complete survey of encounters up to the present time.

Essentially since the late 1940's, the literature has dealt with the description, detection, and damping of the oscillations due to their unwanted nature. Curosiously enough, it is Lord Rayleigh (5) that is most often quoted in explaining the oscillations. In his work <u>The Theory of Sound, Volume II</u>, under the section entitled "Maintenance by Heat of Aerial Vibrations," Lord Rayleigh describes a form of thermally sustained vibrations encountered by glass blowers when blowing a bulb at the end of a long narrow tube. Just as found in cryogenic applications, the tube was communicating between temperature extremes and the tube, itself, certainly must have possessed a large temperature gradient along its wall. Qualitatively Lord Rayleigh states that, as the gas is moved from the cooler regions of the tube to warmer regions, "the adjustment of temperature takes <u>time</u> (italics), and thus the temperature of the air deviates from that of the neighboring parts of the tube, inclining towards the temperature of that part of the tube <u>from</u> (italics) which the air has just come. From this it follows that at the phase of greatest

condensation heat is received by the air and at the phase of greatest rarefaction heat is given up from it, and thus there is a tendency to maintain the vibrations." Lord Rayleigh further points out that the bulb and the great range of temperature is necessary for the maintenance of the vibration. Within cryogenic apparatus construction, low conductivity material such as glass, cupro-nickel, or stainless steels are used for the tubes. Thus, a severe temperature gradient must exist near the cold end of the tube; and the warm end of the tube and associated valving act as the bulb of Lord Rayleigh's high temperature phenomenon.

Extensive experimentation has been conducted to determine the types of materials and geometric configurations which will foster and sustain the pressure oscillations in a room temperatureto cryogenic temperature environment. Clement and Gaffney (2), Ditmars and Furukawa (3), Bannister (1), and Thullen (7) have in their own ways measured frequencies, pressure amplitudes, and devised methods of damping the oscillations. From their experimentions and reports it is evident that the tube diameter, length, and magnitude of the longitudinal temperature gradient in the tube wall are the most important physical characteristics of the apparatus.

Analytical investigations have yet to lead to a satisfactory theoretical explaination of the phenomenon associated with cryogenic apparatus. Norton and Mulenhaupt (4) have produced a computer program to simulate the

pressure oscillations. In their program the continuous temperature gradient was approximated by a large number of constant temperature regions. The program gave encouraging results when compared to experimental measurements of the pressure oscillations. Thullen (7) proposed a lumped parameter model in which he was able to form two dimensionless variables, one containing frequency, friction and temperature gradient information, and the other containing amplitude and heat transfer information.. From this approach the exact behavior of any distributed system can not be predicted; but, it does point out the controlling variables. Both the approaches made by Norton and Mulenhaupt and Thullen have added significantly to the theoretical understanding of thermally sustained pressure oscillations in cryogenic equipment. However, due to the approaches taken, neither theory was able to capture the inherent distributiveness of the system and the continuous nature of the temperature gradient along the tube wall.

It is the intention of this author to define a model for the pressure oscillations which encompasses the important points of previous experimentation, and analyze the model in such a manner as to retain as much distributiveness of the system and continuity of the temperature gradient as possible, while maintaining a manageable theoretical discription of the phenomenon.

II. ANALYSIS

1. The Model.

The model chosen for the theoretical analysis of the thermally sustained pressure oscillations closely approximates the geometric configuration often described in the literature. Common points in these configurations are the closed volume at the outside end of the penetrating tube, the longitudinal temperature gradient along the tube wall, and the free communication of the cold vapor with the open end of the tube. The system for analysis is illustrated in Figure 1.

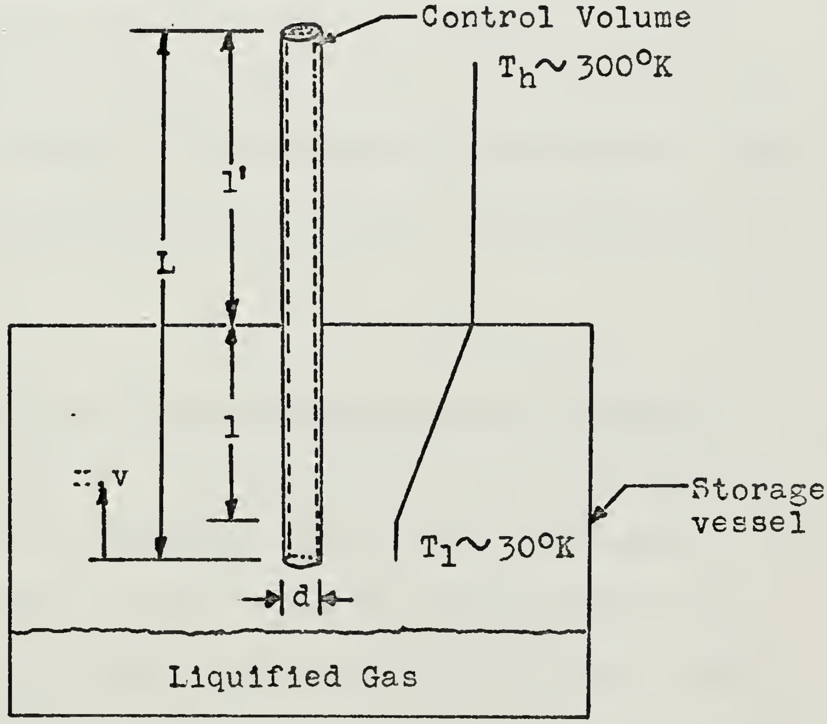


Figure 1. Model For Analysis

It is important that the temperature gradient along the tube wall be of sufficient strength to allow heating of the vapor upon compression and cooling upon expansion. The maximum amplitude of the oscillations is limited due to viscous friction. Thus the model of real systems has all the elements to foster the growth of, and sustain, pressure oscillations. Also, the distributed nature of the system and the continuity of the temperature gradient have been preserved.

Within cryogenic apparatus the oscillations are spontaneous in origin; and, after a period of time, their amplitudes grow to a maximum value. The analysis of this model will assume that a sinusoidally varing steady state condition has been reached within the system. The relationships between time, motion, and heat transfer will be combined to define the pressure as a function of time, physical characteristics of the vapor, and geometric configuration.

2. Derivation of the Time and Pressure Relationship.

When the model is considered as an open system, the oscillations of the vapor within the tube cause energy to be added or lost to the system by virtue of mass entering or exiting the open end of the tube, and by heat transfered to the vapor because of the temperature gradient along the tube wall. The first law of thermodynamics for an open system applies to the control volume

within the closed tube:

$$\frac{dU}{dt} = c_p T_1 \frac{dm}{dt} + \dot{Q} \qquad (1)$$

The time rate of change of mass entering the system across the open end of the tube is given by:

$$\frac{dm}{dt} = \rho A v(t) \qquad (2)$$

where v(t) is the average velocity of the fluid particles at the end of the tube at a given time t.

The perfect gas relations and the definition of internal energy give the rate of change of pressure in terms of the rate of change of internal energy:

$$P = \frac{mRT}{V} \qquad (3)$$

$$U = mc_v T \qquad (4)$$

$$P = \frac{R}{c_v V} U \qquad (5)$$

$$P = CU \qquad (6)$$

where $C = \frac{R}{c_v V}$

Since both pressure and internal energy are functions of time:

$$\frac{dP}{dt} = C\frac{dU}{dt} \qquad (7)$$

When equation (7) is substituted into equation (1), the rate of change of pressure is given as a function of time:

$$\frac{dP}{dt} = C\left[c_p A \rho T_1 v(t) + \dot{Q}(t)\right] \tag{8}$$

Density is not a variable of primary interest; and, over a cycle of the oscillations, its variation is small. Therefore, the following simplifing assumption and substitution is made in equation (8):

$$\rho = \frac{\bar{P}}{R\bar{T}} \tag{9}$$

where $\bar{P}$ is the average pressure at an average temperature, $\bar{T}$, during a cycle:

$$\frac{dP}{dt} = C\left[\frac{c_p \bar{P} A}{R}\left(\frac{T_1}{\bar{T}}\right)v(t) + \dot{Q}(t)\right] \tag{10}$$

When v(t) and $\dot{Q}(t)$ are known, equation (10) can be integrated to give the pressure as a function of time.

Since the usual tube diameters are large in comparison to the thermal and momentum boundary layer thicknesses, the derivation of the velocity term, v(t), and the heat transfer term, $\dot{Q}(t)$, in equation (10) will be carried out in a rectangular co-ordinate system. Appendix A shows the range of tube diameters for which this approach will be valid.

3. Solution of the Velocity Term, $v(t)$.

Schlichting (6) presents the boundary layer solution for the flow of a fluid near an oscillating flat plate. For the purpose of this analysis, Schlichting's solution will be modified to fit the boundary condition that the fluid is sinusoidally oscillating over a stationary flat plate. Accordingly, the velocity distribution in the fluid above the flat plate as a function of height above the plate, y, and time, t, is given by:

$$u(y,t) = u_o e^{-\delta y} \cos (nt - \delta y) - u_o \cos nt \tag{11}$$

where $\delta = \sqrt{\frac{n}{2\nu}}$ is the spacial decay rate of the momentum boundary layer.

The average velocity of the fluid entering the tube is obtained by assuming that the average velocity of the fluid over the flat plate to a given distance is analogous to that entering the tube:

$$v(t) = \frac{\int_0^{y_m} u(y,t)dy}{y_m} \tag{12}$$

where y_m corresponds to the tube radius, d/2, and is sufficiently large so that in equation (12) $e^{-\delta y_m} \ll 1$.

When the average velocity is determined in this manner, the final solution of the velocity term is:

$$v(t) = u_o \left[\left(\frac{1}{2\delta y_m} - 1 \right) \cos nt + \frac{1}{2\delta y_m} \sin nt \right] \tag{13}$$

4. Solution of the Heat Transfer Term, $\dot{Q}$.

The model assumes that the only heat transfer to and from the vapor is in the area of the tube where the temperature gradient exists. The rest of the volume of the tube is assumed to be adiabatic with respect to the vapor. As with the velocity term derivation, the flat plate approach is assumed to be applicable.

To begin, an energy balance is made on a control volume of unit depth positioned above the flat plate. There is a vertical heat flux through the control volume and a horizontal enthalphy flux. Figure 2 illustrates the energy balance.

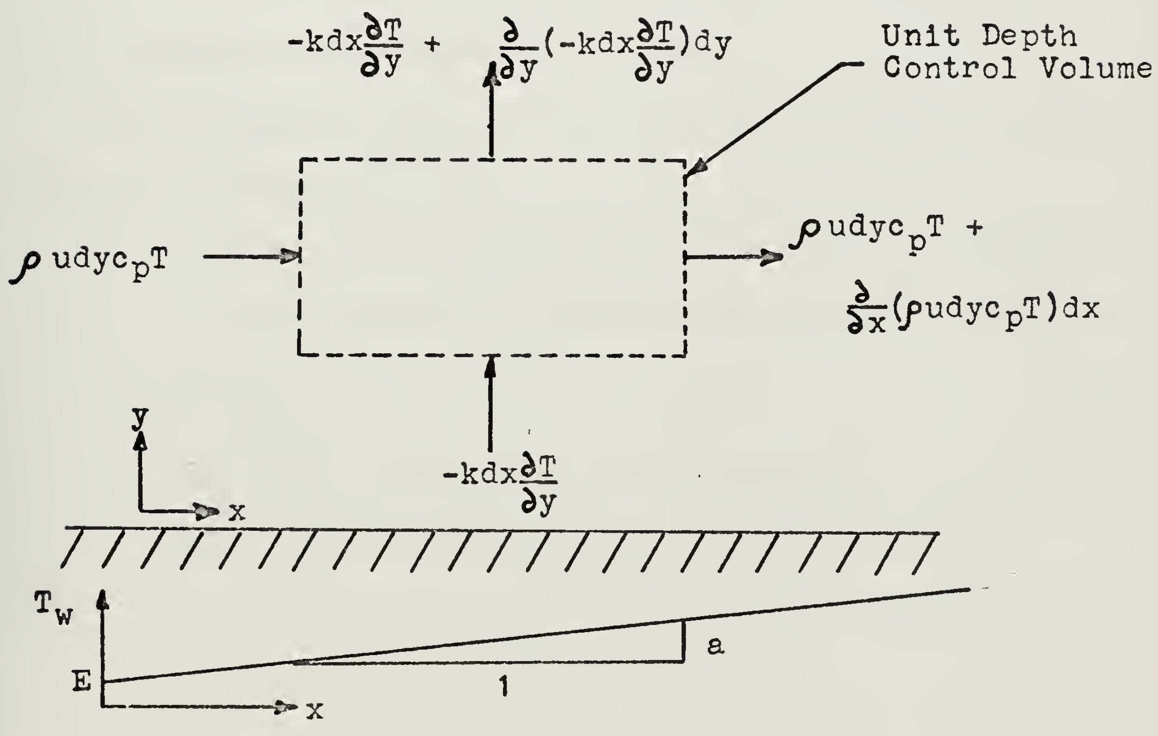


Figure 2. Energy Balance For Heat Transfer Derivation

The energy balance on the control volume yields the following differential equation:

$$\rho c_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} - \rho c_p u(y,t) \frac{\partial T}{\partial x} \tag{14}$$

Equation (14) may be rearranged into a diffusion equation with a velocity type forcing function:

$$\frac{1}{\gamma} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} - u(y,t) \frac{\partial T}{\partial x} \tag{15}$$

where $\gamma = \dfrac{c_p}{c_v}$ and $\alpha = \dfrac{k}{\rho c_p}$.

When equation (15) has been solved for the temperature distribution in the fluid above the flat plate, the temperature gradient at the wall can be found by differentiating the solution with respect to y and setting y equal to zero. This will enable the heat transfer term to be realized as a function of time.

The following assumptions are made in reguard to the solution of equation (16):

a. A linear temperature gradient exists in the wall so that the wall surface temperature is given by:

$$T_w = ax + E \tag{16}$$

b. The same linear temperature gradient as in the wall exists within the gas so that the form of the temperature distribution in the gas above a point on the plate is given by:

$$T(y,t) = f(y,t) + ax \tag{17}$$

c. There is neglible heat conduction in the gas in the x-direction.

d. The velocity distribution is given by equation (11) and the average velocity is given by equation (13).

Equation (17) must also fit the boundary condition that at the wall the wall surface temperature and the gas temperature must be equal.

When equation (17) is substituted into equation (15), the differential equation which determines the temperature distribution above the plate is:

$$\frac{1}{\gamma}\frac{\partial f}{\partial t} = \alpha\frac{\partial^2 f}{\partial y^2} - au(y,t) \qquad (18)$$

where the function $f(y,t)$ is subject to the boundary condition that at $y = 0$, $f(0,t) = E$.

The solution equation (18) is of the form:

$$\begin{aligned} f(y,t) = e^{-\delta y}\, & A_1 \sin(nt - \delta y) + A_2 \sin(nt - \delta y) \\ & + B_1 \sin nt + B_2 \cos nt \\ & + e^{-\xi y}\, D_1 \sin(nt - \xi y) + D_2 \cos(nt - \xi y) \\ & + \text{constant} \end{aligned} \qquad (19)$$

Appendix B shows the determination of constants associated with equation (19).

The complete solution to equation (17) giving the temperature distribution in the gas above the flat plate is:

$$T(y,t) = E + ax + \frac{u_o a\, e^{-\delta y}}{2\alpha\delta^2 - \frac{n}{\gamma}} \sin(nt - \delta y)$$

$$+ \frac{u_o a \gamma}{n} \sin nt$$

$$- \frac{2u_o a \alpha \delta^2 \gamma / n}{2\alpha\delta^2 - \frac{n}{\gamma}} e^{-\xi y} \sin(nt - \xi y) \qquad (20)$$

where $\xi = \sqrt{\frac{n}{2\alpha\gamma}}$, spacial decay rate of thermal boundary layer

$\delta = \sqrt{\frac{n}{2\nu}}$, spacial decay rate of momentum boundary layer.

The heat transfered from the plate to the oscillating vapor is given by:

$$\dot{Q} = -k\frac{\partial T}{\partial y}\Big|_{y=0} A' \qquad (21)$$

When equation (20) is differentiated with respect to y and evaluated at y = 0 and the result substituted into equation (21), the heat transfer term, $\dot{Q}$, is given by:

$$\dot{Q} = \frac{-kaA'u_o\delta}{2\alpha\delta^2 - \frac{n}{\gamma}}\left[\frac{2\alpha\delta\xi\gamma}{n} - 1\right]\left[\sin nt + \cos nt\right] \qquad (22)$$

5. <u>Final Solution of the Pressure-Time Relationship</u>.

With the results of sections 3. and 4. combined with the differential equation of section 2., the final derivation of the gas pressure as a function of time can be determined.

Equation (10) of section 2. is:

$$\frac{dP}{dt} = C\left[\frac{c_p\bar{P}A}{R}\left(\frac{T_1}{\bar{T}}\right)v(t) + \dot{Q}(t)\right]$$

Equation (13) of section 3. is:

$$v(t) = u_o\left[\left(\frac{1}{2\delta y_m} - 1\right)\cos nt + \frac{1}{2\delta y_m}\sin nt\right]$$

Equation (22) of section 4. is:

$$\dot{Q} = \frac{-kaA'\delta u_o}{2\alpha\delta^2 - \frac{n}{\gamma}}\left[2\alpha\delta\xi\frac{\gamma}{n} - 1\right]\left[\sin nt + \cos nt\right]$$

When equations (13) and (22) are substituted into (10) and equation (10) is integrated with respect to time, the gas pressure as a function of time is:

$$P(t) = C\left\{\left(\frac{c_p\bar{P}A}{R}\right)\left(\frac{T_1}{\bar{T}}\right)\left(\frac{u_o}{n}\right)\left[\left(\frac{1}{2\delta y_m} - 1\right)\sin nt - \frac{1}{2\delta y_m}\cos nt\right] - \left(\frac{kaA'\delta}{2\alpha\delta^2 - \frac{n}{\gamma}}\right)\left(\frac{u_o}{n}\right)\left(2\alpha\delta\xi\frac{\gamma}{n} - 1\right)\left(-\cos nt + \sin nt\right)\right\} + P_\infty \qquad (23)$$

where P_∞ is the constant of integration or pressure at time $t = 0^-$.

With the following definitions the compressible volume pressure given by equation (24) can be made non-dimensional and more manageable for practical applications:

Let $N_o = \dfrac{1}{2\delta\, y_m}$

$$N_1 = 2\alpha\delta\xi\frac{\gamma}{n}$$

$$N_2 = 2\alpha\delta^2\frac{\gamma}{n}$$

$$T^{*} = \frac{T_1}{\bar{\bar{T}}}$$

$$H = \frac{c_p \bar{P} A}{\bar{R}}$$

$$B = \frac{k a A' \delta \partial'}{n}$$

Upon the substitution of the above definitions, equation (23) becomes:

$$\frac{P(t)}{(BCu_o/n)} = \left[\frac{HT^{*}}{B}(N_o - 1) - \left(\frac{N_1 - 1}{N_2 - 1}\right)\right] \sin nt - \left[\frac{HT^{*}}{B} N_o - \left(\frac{N_1 - 1}{N_2 - 1}\right)\right] \cos nt \qquad (24)$$

In a likewise manner the average velocity, $v(t)$, can be made non-dimensional:

$$\frac{v(t)}{u_o} = (N_o - 1)\cos nt + N_o \sin nt \qquad (25)$$

III. DISCUSSION

With the pressure history of the warm volume and the motion of the mass taking part in the oscillations now known, a dynamical approach to the system will be used to derive the relationship between the parameters which control the oscillations. The model of the system for this investigation will be similar to that used by Thullen (7) and is shown in Figure 3.

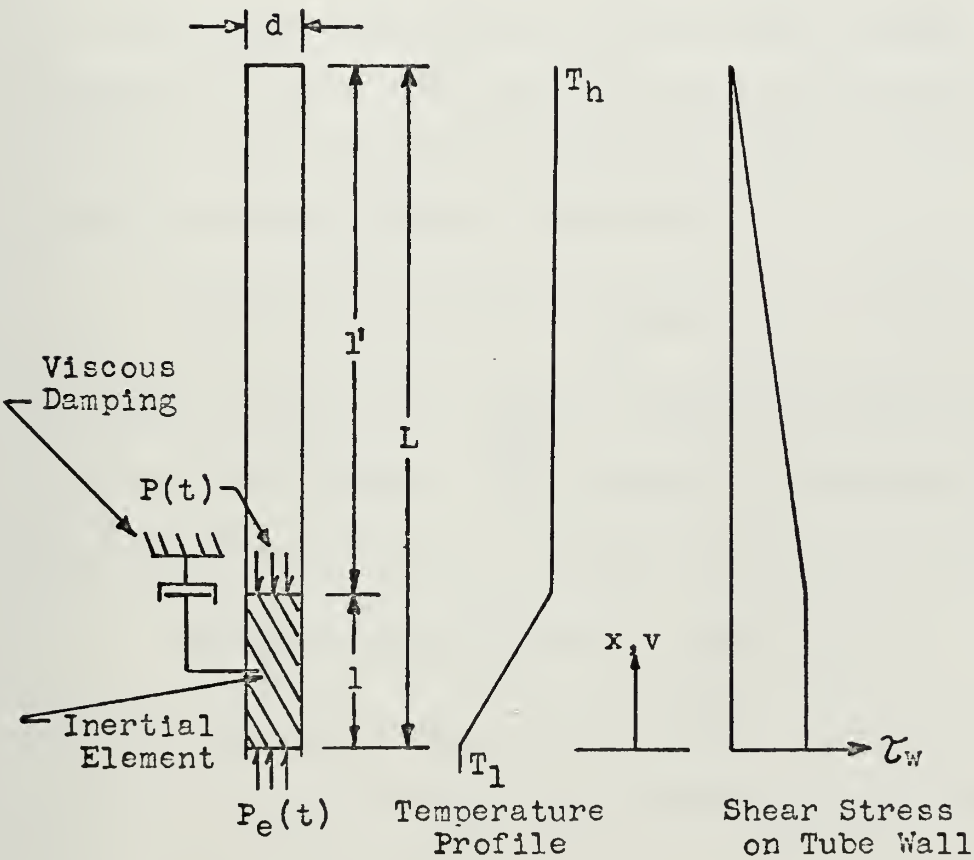


Figure 3. Model For Cyclic Interpretation

Newton's law of motion applies to the inertial element of the model:

$$m\ddot{x} + D\dot{x} - A\left[P(t) - P_e(t)\right] = 0 \tag{26}$$

The transient solution to equation (26) is not of interest and can be avoided by the cyclic integration of equation (26) with respect to displacement. The net work done in accelerating the inertial element through a cycle is zero:

$$\oint m\ddot{x}dx = 0 \tag{27}$$

The two remaining terms of equation (26) are now equated and point out that the cyclic work done on the volume to the left of the inertial mass is equal to the cyclic work done against friction:

$$\oint A\left[P(t) - P_e(t)\right] dx = \oint D\dot{x}dx \tag{28}$$

The identity $dx = \frac{dx}{dt}\,dt = v(t)dt$ is substituted into equation (28) so that the cyclic integration can be performed with respect to time instead of displacement.

$$\oint A\left[P(t) - P_e(t)\right] v(t)dt = \oint Dv^2(t)dt \tag{29}$$

If the exit pressure is constant, $P_e(t) = P_\infty$, the following enequality must be satisfied for the oscillations to be self sustaining:

$$\oint A\left[P(t) - P_\infty\right] v(t)dt \geqq \oint Dv^2(t)dt \tag{30}$$

In substituting equations (24) and (25) into (30) and performing the cyclic integration, the following relationship is established:

$$\frac{ABC}{nD}\left(\frac{N_1 - 1}{N_2 - 1}\right) \gtrapprox (N_o - 1)^2 + N_\delta^2 \tag{31}$$

Thullen's (7) non-dimensional parameter containing frequency, friction, and temperature gradient information is as follows:

$$N_T = \frac{P_o A^2 \gamma}{s} \, b \, \frac{1}{D\omega\pi} \tag{32}$$

Upon substitution of the following definitions of the variables in equatiion (32), the Thullen parameter becomes:

$$P_o = \rho R T_1$$

$$A = \frac{\pi d^2}{4}$$

$$s = \frac{\pi d^2 l'}{4}$$

$$b = \frac{\Delta T}{T_1}$$

$$\omega = n$$

$$N_T = \left[\frac{d^2}{4l'}\right]\left[\rho R \Delta T \gamma\right]\left[\frac{1}{Dn}\right] \tag{33}$$

Upon subsitituion of the actual variables for B and C in the non-dimensional grouping $\frac{ABC}{nD}$ in equation (31) and making the following definitions, this non-dimensional group is seen to contain Thullen's parameter plus additional information:

$$N^* = \frac{ABC}{nD} = \frac{AkaA'\delta\gamma'}{n^2 D}\ \frac{R}{c_v V} \tag{34}$$

$$A = \frac{\pi d^2}{4}$$

$$A' = \pi d l$$

$$V = \frac{\pi d^2}{4}\, l'$$

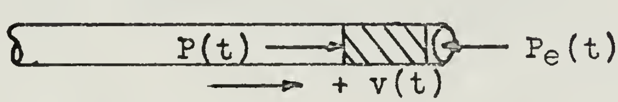
$$\delta = \sqrt{\frac{n}{2\nu}} = \sqrt{\frac{n\rho}{2\mu}} = \rho\sqrt{\frac{n}{2\mu\rho}}$$

$$N^* = \left[\frac{d^2}{4l'}\right]\left[\rho R \Delta T \gamma'\right]\left[\frac{1}{Dn}\right]\left[\frac{4\pi k}{c_v n^{\frac{1}{2}}\, d\sqrt{2\mu\rho}}\right] \tag{35}$$

$$N^* = N_T\left[\frac{4\pi k}{c_v n^{\frac{1}{2}} d\ \sqrt{2\mu\rho}}\right] \tag{36}$$

In order to carryout the cyclic integration of equation (29) with a variable exit pressure, $P_e(t)$, further definition of the exit pressure term and the damping coefficient term, D, are required.

First, the exit pressure is examined:



Outflow: $P_e(t) = P_\infty$

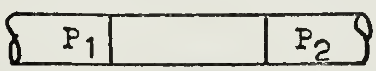
Inflow: $P_e(t) = P_\infty - \frac{1}{2}\rho v^2(t)$

$$\oint AP_e(t)v(t)dt = \int_{\text{inflow } \frac{1}{2} \text{ cycle}} AP_e(t)v(t)dt + \int_{\text{out flow } \frac{1}{2} \text{ cycle}} AP_e(t)v(t)dt \tag{37}$$

The outflow half-cycle contributes nothing to the cyclic integration. In the inflow half-cycle the sign of the velocity is negative. Therefore, the cyclic integration of equation (29) reduces to:

$$\oint A\left[P(t) - P_\infty\right]v(t)dt - \int_{\text{inflow } \frac{1}{2} \text{ cycle}} \tfrac{1}{2}A\rho v^3(t)dt = \oint Dv^2(t)dt \tag{38}$$

Second, the damping coefficient term can be determined through a pipe flow analogy involving shear stress:



$$F = (P_1 - P_2)\, A = \tau_w A_w$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y = 0}$$

The rate of change of velocity with respect to y is found from the velocity equation (11)

$$\left.\frac{\partial u}{\partial y}\right|_{y = 0} = -\delta\, u_0\left[\sin nt + \cos nt\right] \tag{39}$$

The damping coefficient is now defined as:

$$D = \frac{-F}{v(t)} = \frac{A_w \delta \mu u_0\left[\sin nt + \cos nt\right]}{v(t)} \tag{40}$$

The right half of equation (38) now requires the cyclic integration of the following term:

$$\oint \frac{A_w \delta \mu u_0\left[\sin nt + \cos nt\right]}{v(t)}\, v^2(t)dt \tag{41}$$

With the above substitutions for the exit pressure and damping coefficient terms and subsequent cyclic integration of equation (38), the relationship between the parameters which contribute to the sustaining of the pressure oscillations can be established:

$$\frac{ABC\ u_o^2\pi}{n^2}\left(\frac{N_1 - 1}{N_2 - 1}\right) - \frac{2}{3}\frac{\rho A u_o^2}{n}\left[(N_o - 1)^2 + N_o^2\right]^{3/2} =$$

$$\frac{\pi A_w k\ u_o^2}{n}\left[(N_o - 1)^2 + N_o^2\right]^{1/2} \qquad (42)$$

Upon substitution of the following definitions and previous values of B and C, equation (42) takes the form:

$$\phi = \left(\frac{N_1 - 1}{N_2 - 1}\right) = \left(\frac{\sqrt{\frac{\alpha\gamma}{\nu}} - 1}{\frac{\alpha\gamma}{\nu} - 1}\right)$$

$$\psi = (N_o - 1)^2 + N_o^2 = \left(\frac{1}{\delta d} - 1\right)^2 + \left(\frac{1}{\delta d}\right)^2$$

$$u_o = nx_o$$

$A_w = \pi dL'$, where L' is the reduced length $(L - \frac{1}{2}l')$ of the tube since the shear stress is not uniform over the entire length of the tube of Figure 3.

$$\delta = \sqrt{\frac{n}{2\nu}} = \sqrt{\frac{n\rho}{2\mu}}$$

$$\frac{1}{3}\frac{d}{L'}\sqrt{\frac{n}{2\nu}}\,x_o = \frac{k\ T\gamma(\gamma - 1)}{l'L'\mu n^2}\phi\psi^{-2/3} - \psi^{-1/6} \qquad (43)$$

Figure 4. shows the non-dimensional displacement amplitude, $\frac{1}{3}\frac{d}{L'}\sqrt{\frac{n}{2\nu}}x_o$, plotted against the driving force parameter, $\frac{k\Delta T\gamma(\gamma - 1)}{l'L'\mu n^2}$, with lines of constant ψ. The graphical representation of equation (43) gives indication of a given system's performance and how that performance will change if the various parameters are altered. The manner of change suggested follows closely that which has so often been described in literature on thermally sustained pressure oscillations. Due to the lower limit on tube diameter imposed by the theoretical approach taken, the lines of constant ψ in Figure 4. are in aggrement with the theory for $\psi \leqq 1$. Values of $\psi \geqq 1$ are shown in dashed lines for completeness.

Of particular interest is the linear relationship between the displacement amplitude and the slenderness ratio, L'/d. Bannister (1) noted this in his experimentation with oscillations over liquid helium. He concluded that the pressure amplitudes are directly proportional to the slenderness ratio of the gas column. The Bannister data was used to calculate values of ψ and the driving force parameter. The calculated operating points are plotted on Figure 4. Appendix C tabulates the constants and data used. Figure 5. shows the calculated amplitude parameter plotted against the measured maximum pressure amplitude. The actual slenderness ratio of Bannister's tubes is also plotted against the maximum pressure amplitude to show

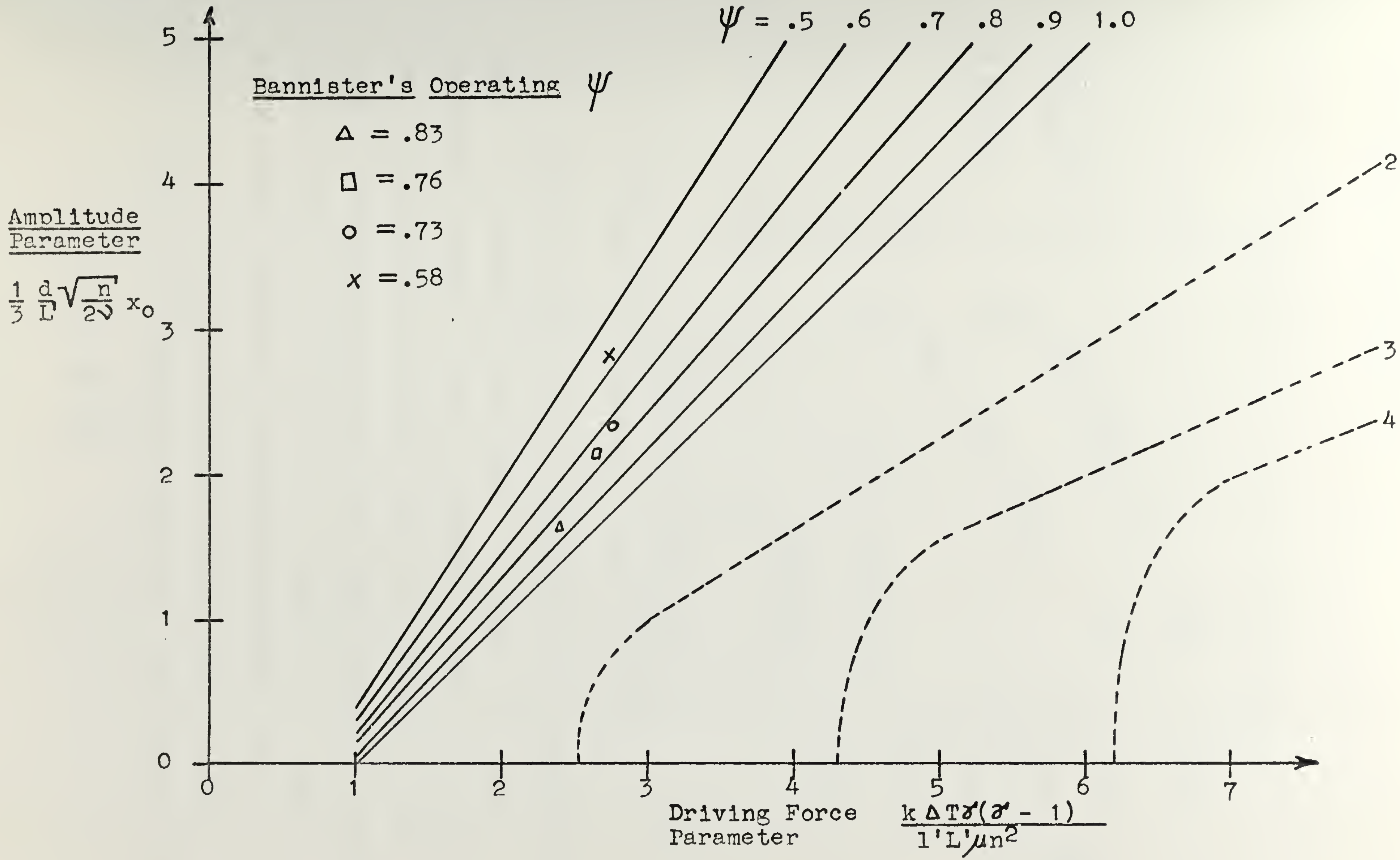


Figure 4. Study of Oscillation Amplitude Parameter verus Driving Force

the linearity comparison.

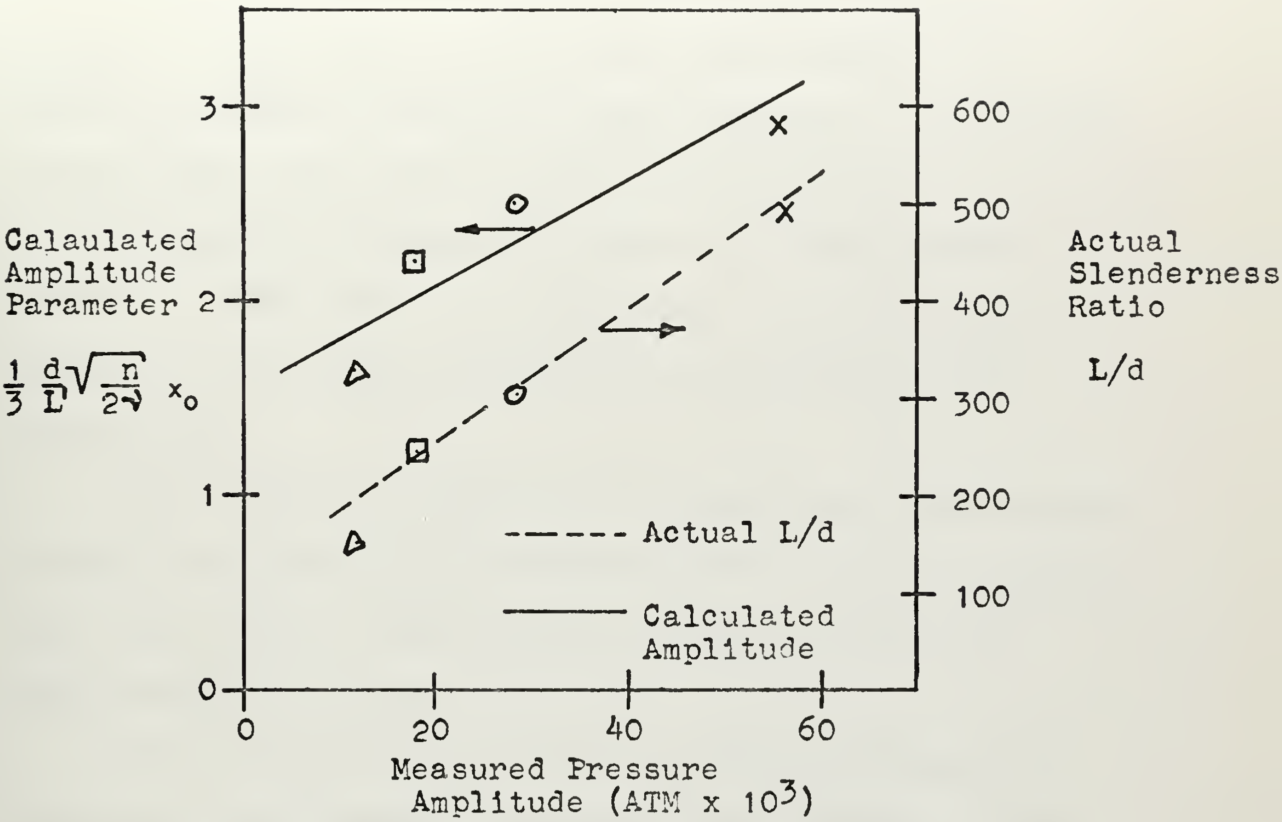


Figure 5. Slenderness Ratio Comparison

Thullen (7) also found a non-dimensional displacement amplitude parameter in his analysis: $x_m/2l$. The characteristic length, l, was the distance between two ideal heat exchangers in his model; and, it was viewed as the distance traveled by the gas after a flow reversal before significant heat transfer begins. However, the length was not well defined for a distributed system. The results of the present analysis indicate that Thullen's "l" might have the following value for a distributed system:

$$l = 3\,\frac{L'}{d}\sqrt{\frac{2\nu}{n}}$$

IV. CONCLUSIONS

A greater understanding of thermally sustained pressure oscillations in liquified gas apparatus is possible through a study of vapor motion and heat transfer interaction. From such a study a time dependent expression for the pressure history of a vapor in a tube can be derived. With cyclic interpretation the relationship among the controling parameters of the system can be established.

When changes are made in the parameters, the behavior of the theoretical relationships appear to follow closely that which occurs when similiar changes are made in experimental apparatus.

The rectangular co-ordinate system from which the motion and heat transfer interaction was derived places a lower bound on the diameter of the tube for which the theory applies. Therefore, an analysis with the same general approach to the motion and heat transfer, but in cylinderical co-ordinates, would be of value. It is not anticipated that this would alter the basic relationship between parameters controling the oscillations; however, the cylinderical system might provide more exact numerical constants. Appendix D suggests an approach to the analysis of the phenomenon of thermally sustained pressure oscillations in a cylinderical co-ordinate system.

APPENDIX

APPENDIX A

In order to show that the flat plate analogy is valid for the results desired, it is sufficient to show that the spacial decay rate is small in comparison to the tube radius.

$$\delta = \frac{n}{2\nu}$$

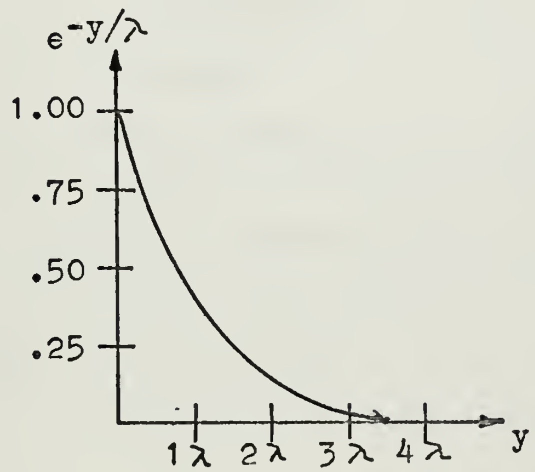
Let $\delta y = y/\lambda$

$$\lambda = \sqrt{\frac{2\nu}{n}}$$

5×10^{-4} ft^2/sec

50 rad/sec

then $\lambda = 0.0535$ in



Hence, tubes with radii $>$ 0.125 in. will be in the range for which the theoretical approach is valid.

APPENDIX B

The determination of the constants associated with equation (19) which is the solution to the differential equation (18) is as follows:

Equation (18): $\frac{1}{\gamma}\frac{\partial f}{\partial t} = \alpha\frac{\partial^2 f}{\partial y^2} - au(y,t)$

where $u(y,t) = u_o e^{-\delta y}\cos(nt - \delta y) - u_o \cos nt$

and $f(y,t)$ must satisfy the boundary condition: $f(0,t) = E$.

Equation (19):

$$\begin{aligned} f(y,t) = e^{-\delta y}&\left[A_1\sin(nt - \delta y) + A_2\cos(nt - \delta y)\right] \\ &+ B_1\sin nt + B_2\cos nt \\ &+ e^{-\rho y}\left[D_1\sin(nt - \rho y) + D_2\cos(nt - \rho y)\right] \\ &+ \text{constant.} \end{aligned}$$

Thus the form of the function f is: $f(y,t) = f_1 + f_2 + \text{Constant}$

Seek the solution of equation (18) by f_1:

$$\begin{aligned} \frac{\partial f_1}{\partial t} = e^{-\delta y}&\left[A_1 n\cos(nt - \delta y) - A_2 n\sin(nt - \delta y)\right] \\ &+ B_1 n\cos nt + B_2 n\sin nt. \end{aligned}$$

$$\frac{\partial^2 f_1}{\partial y^2} = 2\delta^2 e^{-\delta y}\left[-A_2\sin(nt - \delta y) + A_1\cos(nt - \delta y)\right]$$

Upon substitution of these partial derivatives into equation (18) and with like coefficients of the sine and cosine terms equated:

$$A_1 = \frac{u_o a}{(2\alpha\delta^2 - \frac{n}{\gamma})}$$

$$A_2(2\alpha\delta^2 - \frac{n}{\ }) = 0, \qquad A_2 = 0$$

$$B_1 = u_0 a\frac{\gamma}{n}$$

$$B_2 = 0$$

$$f_1(y,t) = \frac{u_0 a e^{-\delta y}}{2\alpha\delta^2 - \frac{n}{\gamma}}\sin(nt - \delta y) + u_0 a\frac{\gamma}{n}\cos nt$$

However, f_1, alone, will not fit the boundary condition. Hence, an additional function must be added to f_1 and that function must be a solution to the homogeneous part of equation (18): $\frac{1}{\gamma}\frac{\partial f}{\partial t} = \alpha\frac{\partial^2 f}{\partial y^2}$

$$f_2(y,t) = e^{-\xi y}\left[D_1\sin(nt - \xi y) + D_2\cos(nt - \xi y)\right] + \text{constant}$$

$$\frac{\partial f_2}{\partial t} = ne^{-\xi y}\left[-D_2\sin(nt - \xi y) + D_1\cos(nt - \xi y)\right]$$

$$\frac{\partial^2 f_2}{\partial t^2} = 2\xi^2 e^{-\ y}\left[-D_2\sin(nt - \xi y) + D_1\cos(nt - \xi y)\right]$$

Upon the substitution of the partial derivatives into the homogenous differential equation and with like coefficients equated, f_2 becomes:

$$D_2(2\alpha\xi^2 - \frac{n}{\gamma}) = 0 \qquad D_1(2\alpha\xi^2 - \frac{n}{\gamma}) = 0$$

If $D_1 \neq 0$ and $D_2 \neq 0$, then $(2\alpha\xi^2 - \frac{n}{\gamma}) = 0$

Therefore, $\xi = \sqrt{\frac{n}{2\alpha\gamma}}$

With the boundary condition now applied to f(y,t), the coeffients D_1, D_2 and the "constant" of f_2 are determined:

$$\text{constant} + \left\{ \left[\frac{2u_o a\alpha\delta^2 \frac{\gamma}{n}}{2\alpha\delta^2 - \frac{n}{\gamma}} \right] + D_1 \right\} \sin nt + D_2 \cos nt = E$$

Therefore,

$$\text{constant} = E$$

$$D_2 = 0$$

$$D_1 = -\frac{2u\ a\alpha\delta^2 \frac{\gamma}{n}}{2\alpha\delta^2 - \frac{n}{\gamma}}$$

$$f_2(y,t) = e^{-\rho y} \left(\frac{-2u\ a\alpha\delta^2 \frac{\gamma}{n}}{2\alpha\delta^2 - \frac{n}{\gamma}} \right) \sin nt + E$$

APPENDIX C

The following constants for helium gas (-200°F) were used in the Bannister data fit of equation (43):

$\alpha = 5.7 \times 10^{-4}$ ft²/sec

$\mu = 8.3 \times 10^{-6}$ lbm/ft-sec

$\nu = 4.0 \times 10^{-4}$ ft²/sec

$k = 2.2 \times 10^{-5}$ BTU/sec-ft-°F

$\gamma = 1.67$

Data from Bannister's experimentation:

	n (cps)	d (cm)	L (cm)	P (atm x 10^3)	ΔT (°F)
×	33.0	.287	149	56	480
o	33.4	.482	149	28	480
□	33.4	.605	149	18	480
Δ	22.7	1.240	213	12	480

Estimated from apparatus geometric configuration:

	L' (cm)	l' (cm)
×	105	89
o	105	89
□	105	89
Δ	105	89

APPENDIX D

The cylinderical co-ordinate analysis will proceed with analogous steps to those performed in rectangular co-ordinates. Figure 5. illustrates the system for analysis. Schlichting's solution for the velocity distribution of an oscillating flow through a pipe is:

Navier-Stokes Equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Assume that the pressure gradient is caused by a harmonically moving piston:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = K\cos nt$$

Use complex notation and attribute physical significance only to the real part:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = Ke^{int}$$

Assume the velocity function has the form:

$$u(r,t) = f(r)e^{int}$$

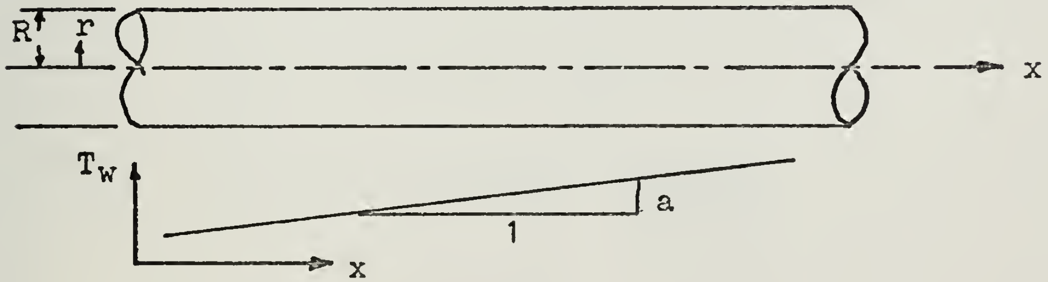


Figure 6. Cylinderical Co-ordinate Model

The solution to the Navier-Stokes equation which gives the velocity distribution within the pipe is:

$$u(r,t) = -i\frac{K}{n}e^{int}\left[1 - \frac{J_o(r\delta)}{J_o(R\delta)}\right]$$

where $\delta = \sqrt{\frac{-in}{\nu}}$

The temperature distribution in the pipe is assumed to have the following form:

$$T(r,t) = f(r,t) + ax$$

Boundary condition for the temperarure distribution is:

$$T(R,t) = T_W = E + ax$$

$$f(R,t) = E$$

The differential equation resulting from energy considerations is:

$$\frac{1}{\gamma}\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right]\right] - au(r,t)$$

The ultimate differential equation to be solved is:

$$\frac{1}{\gamma}\frac{\partial f}{\partial t} = \alpha\left[\frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right] - au(r,t)$$

Assume the solution of the differential equation if of the form:

$$f(r,t) = f_1(r,t) + f_2(r,t) + C$$

where,

$$f_1(r,t) = -iAe^{int}\left[1 - \frac{J_o(r\delta)}{J_o(R\delta)}\right]$$

$$f_2(r,t) = -iBe^{int}\left[1 - \frac{J_o(r\xi)}{J_o(R\xi)}\right] + C, \text{ where } \xi = \sqrt{\frac{-in}{2\alpha\gamma}}$$

The solution is:

$$f_1(r,t) = -\frac{aK}{n}\frac{1}{\left(-i\alpha\delta^2 - \frac{n}{\gamma}\right)}\left[1 - \frac{J_0(r\delta)}{J_0(R\delta)}\right] e^{int}$$

$$f_2(r,t) = \frac{aK}{n}\frac{\alpha\delta^2\frac{\gamma}{n}}{\left(-i\alpha\delta^2 - \frac{n}{\gamma}\right)}\left[1 - \frac{J_0(r\rho)}{J_0(R\rho)}\right] e^{int} + E$$

The temperature distribution within the pipe is:

$$T(r,t) = \left[-\frac{\beta}{\Delta}\left[1 - \frac{J_0(r\delta)}{J_0(R\delta)}\right] + \frac{\beta}{\Delta}\theta\left[1 - \frac{J_0(r\rho)}{J_0(R\rho)}\right]\right] e^{int}$$

$$+ E + ax \text{ , where } \beta = \frac{aK}{n} \text{ , } \Delta = -i\alpha\delta^2 - \frac{n}{\gamma}, \ \theta = \frac{\alpha\delta^2\gamma}{n}$$

The heat transfer from the wall surface to the vapor in the pipe is:

$$\dot{Q} = -k\ 2\pi r l \left.\frac{\partial T}{\partial r}\right|_{r = R}$$

Therefore, the heat transfer term sought is:

$$\dot{Q} = -2\pi k l R\,\frac{\beta}{\Delta}\left[-\delta\frac{J_1(R\delta)}{J_0(R\delta)} + \theta\rho\frac{J_1(R\rho)}{J_0(R\rho)}\right] e^{int}$$

The average velocity of the vapor entering the pipe is:

$$v(t) = \frac{\int_0^R 2\pi r u(r,t)dr}{\pi R^2}$$

$$v(t) = -i\frac{K}{n}\left[1 - \frac{2}{R\delta}\frac{J_0(R\delta)}{J_1(R\delta)}\right] e^{int}$$

Now both the velocity term and heat transfer terms are functions of time and can be substituted into equation (10) of section II.2.:

$$\frac{dP}{dt} = C\left[\frac{c_p\bar{P}A}{R}\frac{T_1}{T}v(t) + \dot{Q}(t)\right]$$

Upon substitution and integration with respect to time, and attaching significance to the real part of the result, the pressure as a function of time is:

$$P(t) - P_\infty = \frac{C}{n}\left\{\left[\frac{c_p\bar{P}A}{R}\left(\frac{T}{\bar{T}}\right)\left(-\frac{K}{n}\right)\left[1 - \frac{2}{R\delta}\frac{J_1(R\delta)}{J_0(R\delta)}\right]\right]\cos nt\right.$$

$$- 2\,klR\,\frac{aK}{n}\,\frac{\left(-\alpha\delta^2\right)}{\left(-\left(\alpha\delta^2\right)^2+\left(\frac{n}{\gamma}\right)^2\right)}\left[-\delta\frac{J_1(R\xi)}{J_0(R\xi)}\right.$$

$$\left. + \theta\xi\frac{J_1(R\xi)}{J_0(R\xi)}\right]\sin nt$$

The average velocity has the following form when returned to real notation:

$$v(t) = \frac{K}{n}\left[1 - \frac{2}{R\delta}\frac{J_1(R\delta)}{J_0(R\delta)}\right]\sin nt$$

The above pressure history of the compressible volume and the average velocity has gouping of parameters which appear similar to those found in the rectangular co-ordinate analysis.

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